

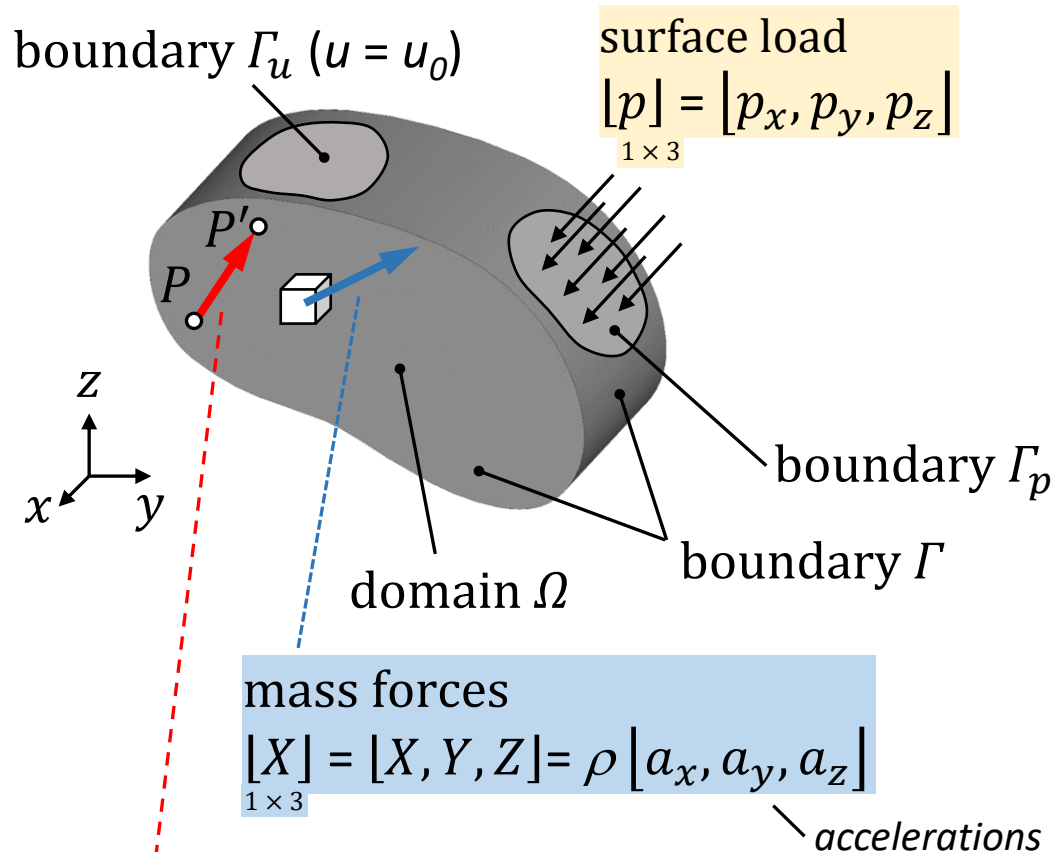


Finite element method (FEM1)

Lecture 2A. The boundary value problem of solid mechanics
in the FEM approach

03.2026

Boundary value problem of solid body mechanics



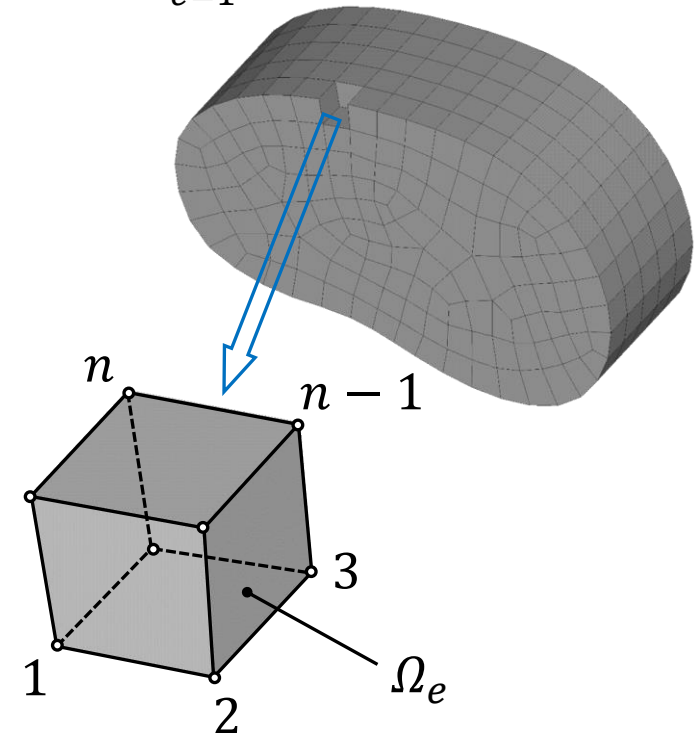
UNKNOWN FUNCTION

Displacement vector $\{u\}_{3 \times 1} = \begin{Bmatrix} u(x, y, z) \\ v(x, y, z) \\ w(x, y, z) \end{Bmatrix}$

FE model

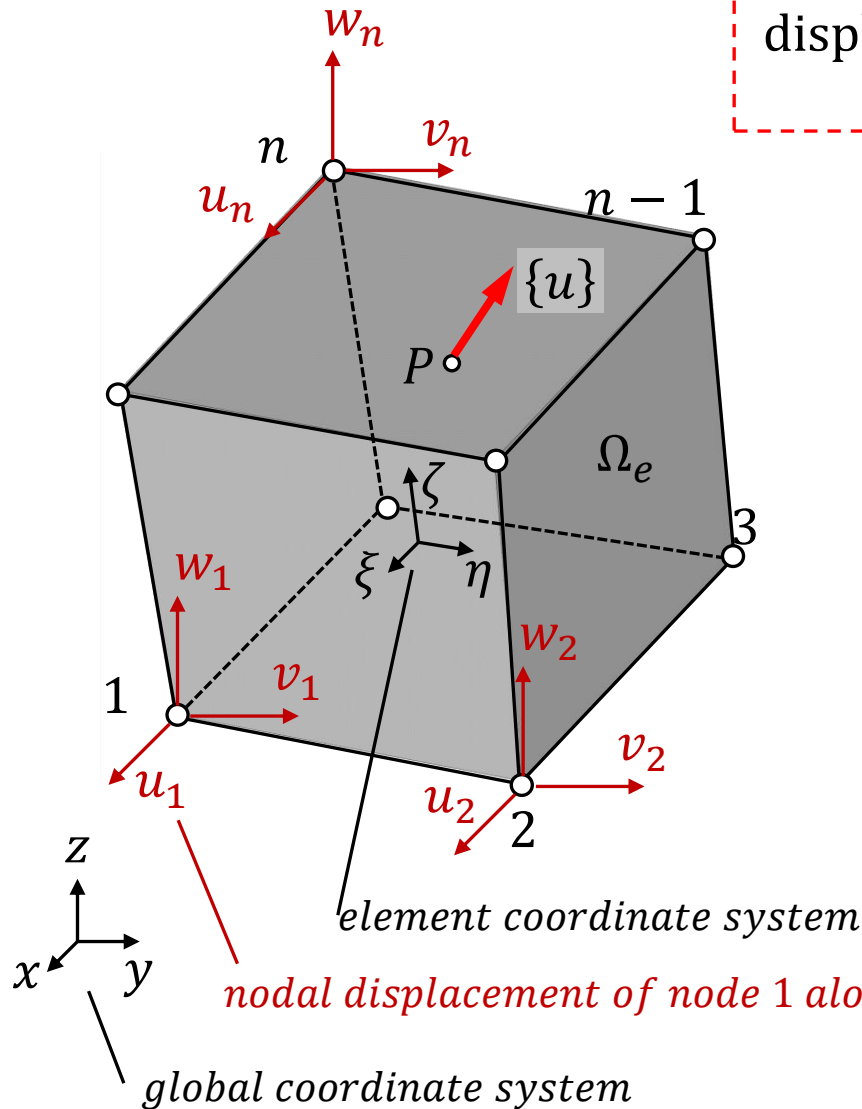
NOE – no. of FEs
 NON – no. of nodes

$$\Omega = \sum_{e=1}^{NOE} \Omega_e \text{ and } \Omega_i \cap \Omega_j = 0 \text{ for } i \neq j$$



Finite element with n - nodes

Nodal approximation inside the finite element with n - nodes



$$\text{displacement vector } \{u\} = [N(\xi, \eta, \zeta)] \{q\}_e$$

3×1 $3 \times n_e$ $n_e \times 1$

$[N(\xi, \eta, \zeta)]$ – matrix of shape functions
 $3 \times n_e$

$$n_e = n \cdot n_p$$

n_e – no. of degrees of freedom in FE

n_p – no. of degrees of freedom per node

$$\{q\}_e = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ \vdots \\ u_n \\ v_n \\ w_n \end{Bmatrix}_e \quad \text{– local vector of nodal parameters}$$

$n_e \times 1$

Matrix of shape functions

Shape function matrix:

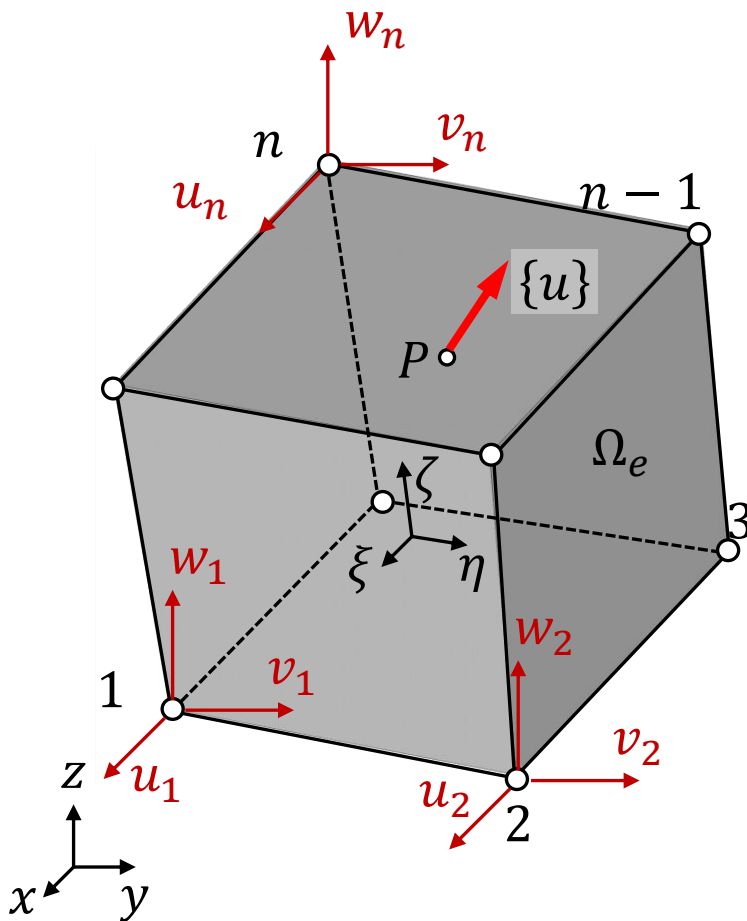
$$[N(\xi, \eta, \zeta)] = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_n & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \dots & 0 & N_n \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_n \end{bmatrix}$$

$3 \times n_e$

Vector of element nodal parameters:

$$\{q\}_e = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ \vdots \\ u_n \\ v_n \\ w_n \end{Bmatrix}_e$$

$n_e \times 1$



Nodal approximation:

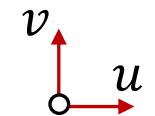
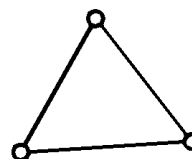
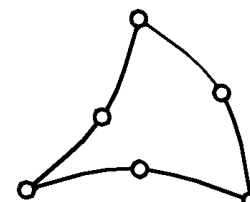
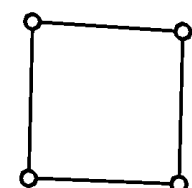
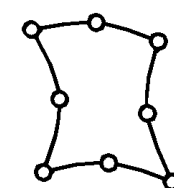
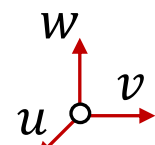
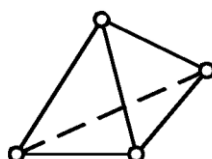
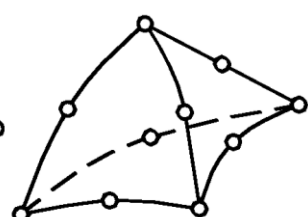
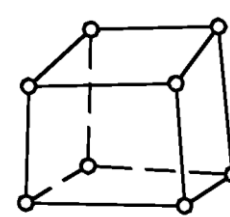
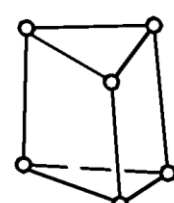
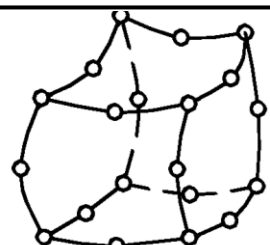
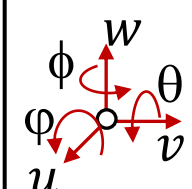
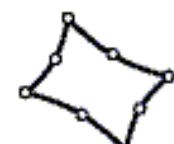
$$\{u\} = [N(\xi, \eta, \zeta)] \{q\}_e$$

$3 \times 1 \quad 3 \times n_e \quad n_e \times 1$

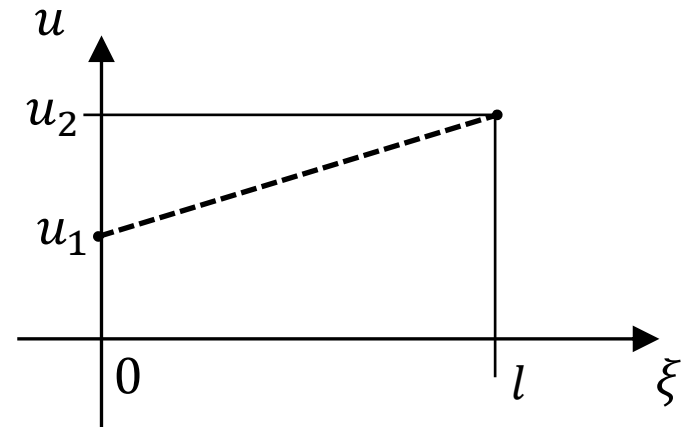
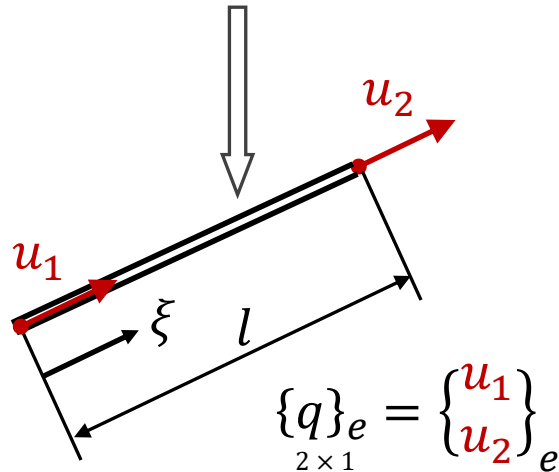
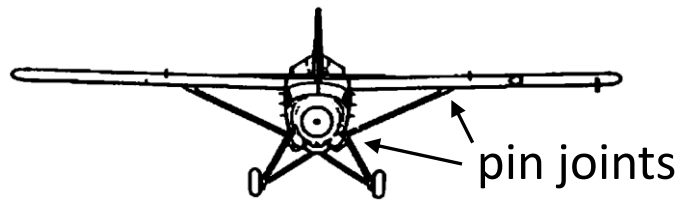


$$\begin{aligned} u &= N_1 \cdot u_1 + N_2 \cdot u_2 + \dots + N_n \cdot u_n \\ v &= N_1 \cdot v_1 + N_2 \cdot v_2 + \dots + N_n \cdot v_n \\ w &= N_1 \cdot w_1 + N_2 \cdot w_2 + \dots + N_n \cdot w_n \end{aligned}$$

Examples of finite elements

Type	n_e – number of degrees of freedom in FE				
rods					
2D 	 6	 12	 8	 16	
3D 	 12	 30	 24	 18	 60
shell 	 24	 48			

Example 1: shape functions for a finite element representing a strut



linear function:

$$u(\xi) = \frac{u_2 - u_1}{l} \xi + u_1$$

$$\begin{aligned} u(\xi) &= \frac{u_2 - u_1}{l} \xi + u_1 = \frac{u_2}{l} \xi - \frac{u_1}{l} \xi + u_1 = \left(1 - \frac{\xi}{l}\right) u_1 + \frac{\xi}{l} u_2 = \\ &= N_1(\xi) \cdot u_1 + N_2(\xi) \cdot u_2 = [N_1, N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_e = [N(\xi)] \{q\}_e \end{aligned}$$

$\quad \quad \quad 1 \times 2 \quad 2 \times 1$

shape functions:

$$N_1(\xi) = 1 - \frac{\xi}{l}$$

;

$$N_2(\xi) = \frac{\xi}{l}$$

Strain components

normal strains:

$$\epsilon_x = \frac{(A'B')_x - AB}{AB} = \frac{(dx + u + \frac{\partial u}{\partial x} dx - u) - dx}{dx} = \frac{\partial u}{\partial x}$$

$$\epsilon_y = \frac{\partial v}{\partial y} ; \quad \epsilon_z = \frac{\partial w}{\partial z}$$

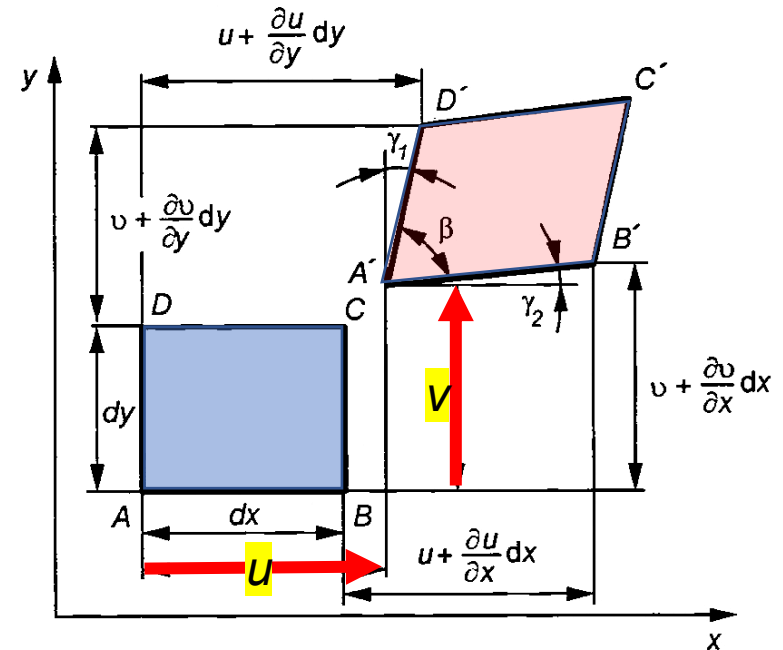
shear strains:

$$\gamma_{xy} = \frac{\pi}{2} - \beta = \gamma_1 + \gamma_2$$

$$\gamma_1 \cong \tan \gamma_1 = \frac{(A'D')_x}{(A'D')_y} = \frac{u + \frac{\partial u}{\partial y} dy - u}{dy + v + \frac{\partial v}{\partial y} dy - v} = \frac{\frac{\partial u}{\partial y}}{1 + \frac{\partial v}{\partial y}} = \frac{\frac{\partial u}{\partial y}}{1 + \epsilon_y} = \frac{\partial u}{\partial y}$$

$$\gamma_2 \cong \frac{\partial v}{\partial x} \rightarrow \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} ; \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} ; \quad \gamma_{ij} = \gamma_{ji}$$



small deformations : $\epsilon_y \ll 1$

Strain tensor. Vector of strain components

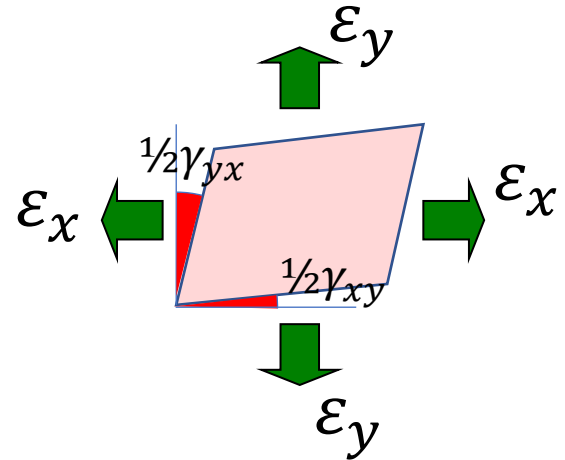
strain tensor:

$$\underset{3 \times 3}{\boldsymbol{\varepsilon}} = \begin{bmatrix} \varepsilon_x & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{yx}/2 & \varepsilon_y & \gamma_{yz}/2 \\ \gamma_{zx}/2 & \gamma_{zy}/2 & \varepsilon_z \end{bmatrix}$$

vector of strain components:

$$\underset{6 \times 1}{\{\boldsymbol{\varepsilon}\}} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \underset{6 \times 3}{[R]} \underset{3 \times 1}{\{u\}} \quad ; \quad \underset{1 \times 6}{[\boldsymbol{\varepsilon}]} = \underset{1 \times 3}{[u]} \underset{3 \times 6}{[R]^T}$$

$\uparrow$
gradient matrix

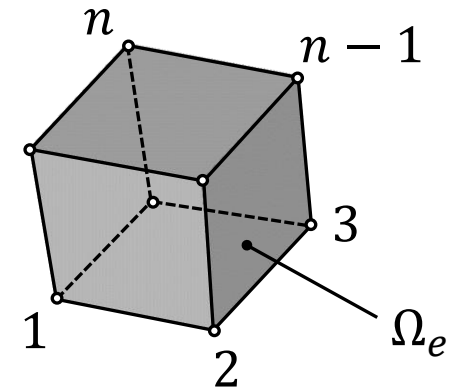


Strain – displacement matrix of a finite element

nodal approximation in a finite element:

$$\{u\} = [N(\xi, \eta, \zeta)]\{q\}_e$$

3×1 $3 \times n_e$ $n_e \times 1$



vector of strain components in a finite element:

$$\{\varepsilon\} = [R]\{u\} = [R][N]\{q\}_e = [B]\{q\}_e$$

6×1 6×3 3×1 6×3 $3 \times n_e$ $n_e \times 1$ $6 \times n_e$ $n_e \times 1$

$$[\varepsilon] = [q]_e [B]^T$$

1×6 $1 \times n_e$ $n_e \times 6$

$$[B] = [R][N] \text{ — strain-displacement matrix}$$

$6 \times n_e$ 6×3 $3 \times n_e$

Stress components

normal stresses:

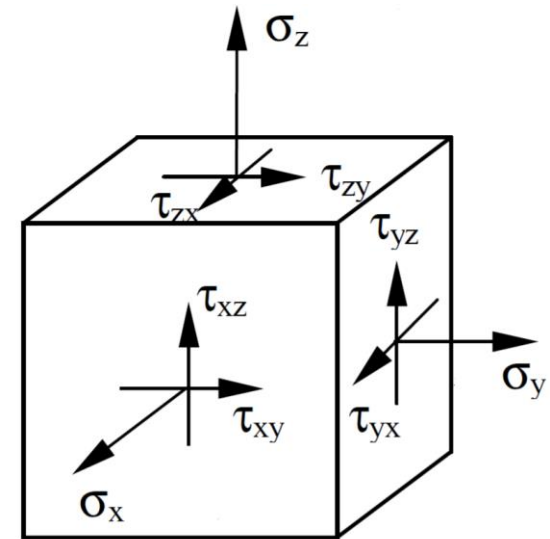
$$\sigma_x \ ; \ \sigma_y \ ; \ \sigma_z$$

positive value - tension, negative value - compression

shear stress components:

$$\tau_{xy} \ ; \ \tau_{yz} \ ; \ \tau_{zx} \ ; \ \tau_{ij} = \tau_{ji}$$

equivalent stresses:



Von Mises stress:

$$\sigma_{EQV} = \sqrt{\frac{1}{2} \left((\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 \right) + 3(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}$$

Tresca stress:

$$\sigma_{INT} = \sigma_1 - \sigma_3 = 2\tau_{max}$$

*the first
principal stress*

*the third
principal stress*

maximum shear stress

Stress tensor. Vector of stress components

stress tensor:

$$\underset{3 \times 3}{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \equiv \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

in (x, y, z) coordinate system

in the principal coordinate system

vector of stress components:

$$\underset{6 \times 1}{\{\boldsymbol{\sigma}\}} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

Constitutive matrix

linear isotropic material (Hooke's law):

$$\underbrace{\{\sigma\}}_{6 \times 1} = \underbrace{[D]}_{6 \times 6} \underbrace{\{\varepsilon\}}_{6 \times 1}$$



constitutive matrix:

$$[D]_{6 \times 6} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-\nu \end{bmatrix}$$

E – Young's modulus,

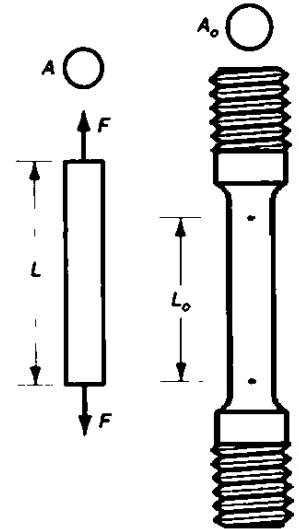
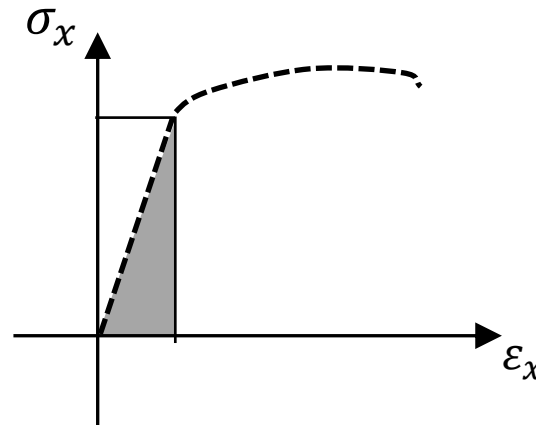
ν – Poisson's ratio

Example 2: uniaxial tensile test

$$\sigma_x = \frac{F}{A_0} \quad ; \quad \varepsilon_x = \frac{L-L_0}{L_0} \quad ; \quad \varepsilon_y = \varepsilon_z = \varepsilon_T$$

elastic strain Energy: $U = \frac{1}{2} \sigma_x \varepsilon_x A_0 L_0$

$$\begin{matrix} \{\sigma\} \\ 6 \times 1 \end{matrix} = \begin{matrix} [D] \\ 6 \times 6 \end{matrix} \begin{matrix} \{\varepsilon\} \\ 6 \times 1 \end{matrix}$$



$$\begin{Bmatrix} \sigma_x \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_T \\ \varepsilon_T \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

2nd equation: $0 = \frac{E}{(1+\nu)(1-2\nu)} (\nu \varepsilon_x + (1-\nu) \varepsilon_T + \nu \varepsilon_T) \rightarrow \boxed{\varepsilon_T = -\nu \varepsilon_x}$

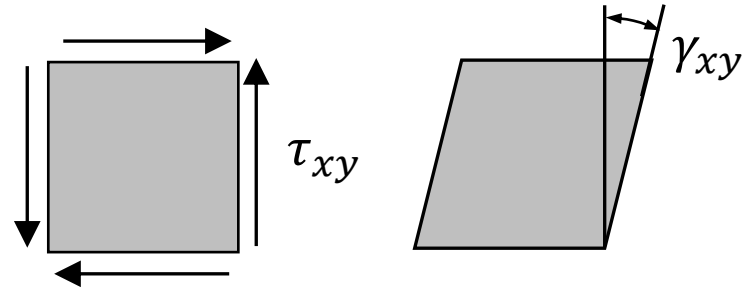
1st equation:

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)} ((1-\nu) \varepsilon_x + \nu \varepsilon_T + \nu \varepsilon_T) = \frac{E}{(1-\nu-2\nu^2)} ((1-\nu) \varepsilon_x - \nu^2 \varepsilon_x - \nu^2 \varepsilon_x) \rightarrow \boxed{\sigma_x = E \varepsilon_x}$$

Example 3: pure shear

$$\tau_{xy} \ ; \ \gamma_{xy}$$

$$\underbrace{\{\sigma\}}_{6 \times 1} = \underbrace{[D]}_{6 \times 6} \underbrace{\{\varepsilon\}}_{6 \times 1}$$



$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \tau_{xy} \\ 0 \\ 0 \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-\nu \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \gamma_{xy} \\ 0 \\ 0 \end{Bmatrix}$$

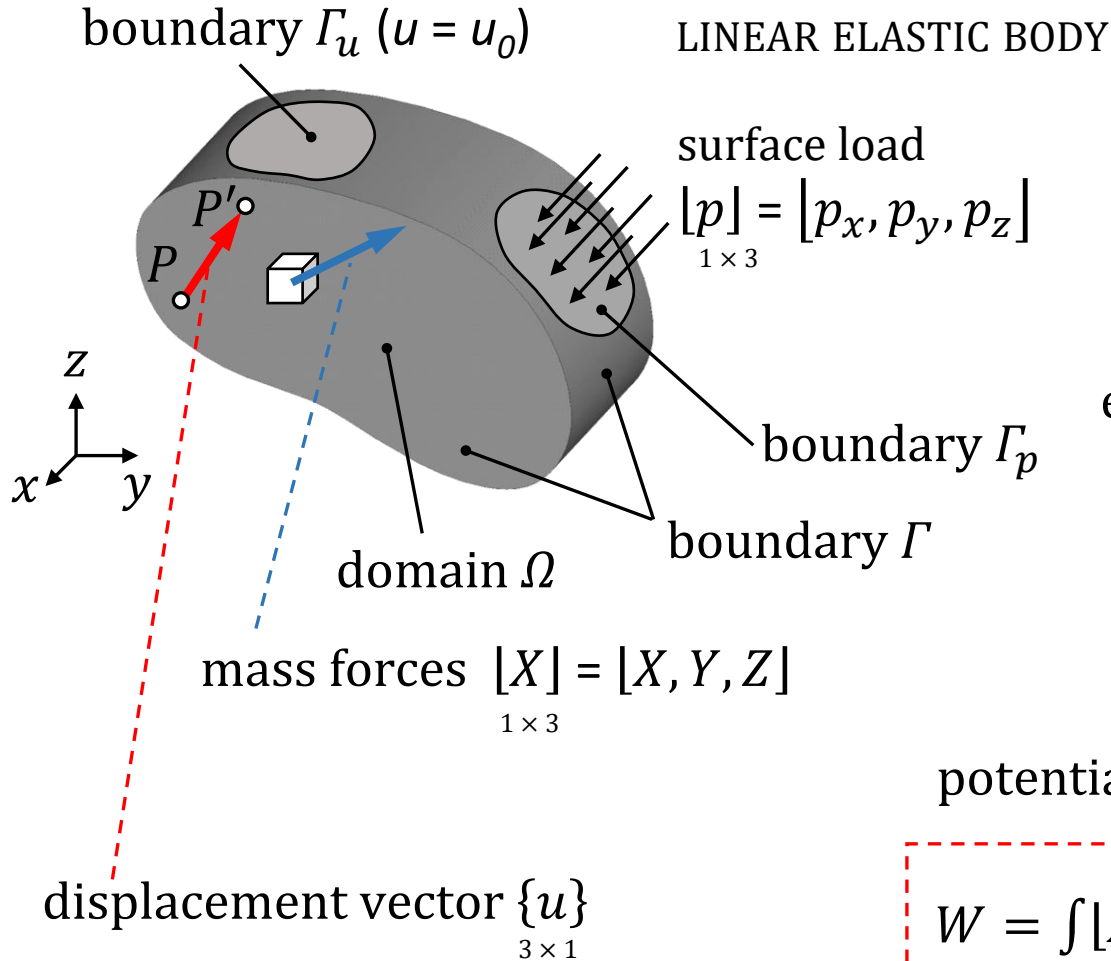
4th equation:

$$\tau_{xy} = \frac{E}{(1+\nu)(1-2\nu)} (0.5 - \nu) \gamma_{xy} = \frac{E}{2(1+\nu)(0.5-\nu)} (0.5 - \nu) \gamma_{xy} = \frac{E}{2(1+\nu)} \gamma_{xy} \rightarrow$$

$$\tau_{xy} = G \gamma_{xy}$$

$$G = \frac{E}{2(1+\nu)} - \text{Kirchhoff's modulus (shear modulus)}$$

Elastic strain energy. Potential energy of loading



elastic strain energy:

$$U = \frac{1}{2} \int_{\Omega} [\varepsilon] \{\sigma\} d\Omega$$

$\Omega \quad 1 \times 6 \quad 6 \times 1$

potential energy of loading:

$$W = \int_{\Omega} [X] \{u\} d\Omega + \int_{\Gamma_p} [p] \{u\} d\Gamma_p$$

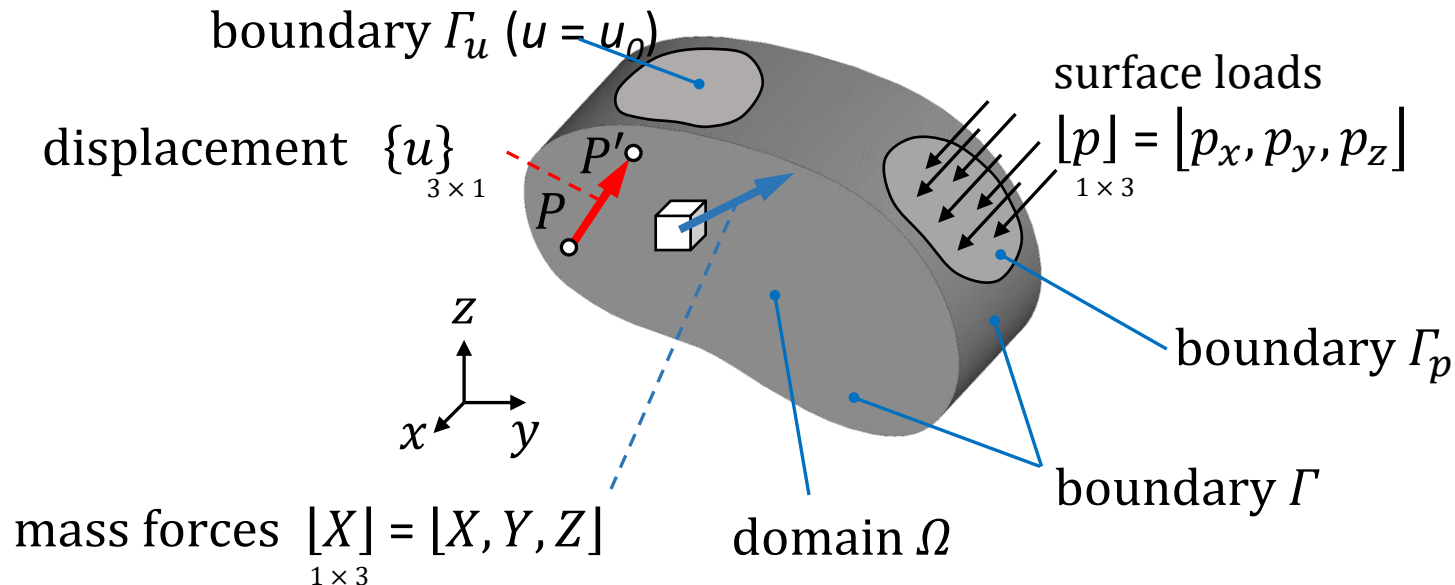
$\Omega \quad 1 \times 3 \quad 3 \times 1 \quad \Gamma_p \quad 1 \times 3 \quad 3 \times 1$

Minimum total potential energy principle

The total potential energy: $V = U - W$

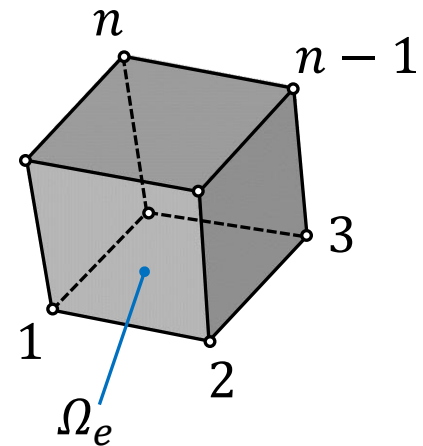
The displacement field $\{u\}$ that represents solution of the problem fulfils displacement boundary conditions on Γ_u and minimizes the total potential energy V .

$$V \rightarrow \min$$



Elastic strain energy in a finite element. Local stiffness matrix

$\{q\}_e$ - local vector of nodal parameters
 $n_e \times 1$



elastic strain energy in a finite element:

$$U_e = \frac{1}{2} \int_{\Omega_e} [\varepsilon] \{\sigma\} d\Omega_e = \frac{1}{2} [q]_e \int_{\Omega_e} [B]^T [D] [B] d\Omega_e \{q\}_e = \frac{1}{2} [q]_e [k]_e \{q\}_e$$

$1 \times 6 \quad 6 \times 1 \quad 1 \times n_e \quad \Omega_e \quad n_e \times 6 \quad 6 \times 6 \quad 6 \times n_e \quad n_e \times 1 \quad 1 \times n_e \quad n_e \times n_e \quad n_e \times 1$

$$[\varepsilon] = [q]_e [B]^T \quad \{\sigma\} = [D] \{\varepsilon\} \quad \{\varepsilon\} = [B] \{q\}_e$$

$1 \times 6 \quad 1 \times n_e \quad n_e \times 6 \quad 6 \times 1 \quad 6 \times 6 \quad 6 \times 1 \quad 6 \times 1 \quad 6 \times n_e \quad n_e \times 1$

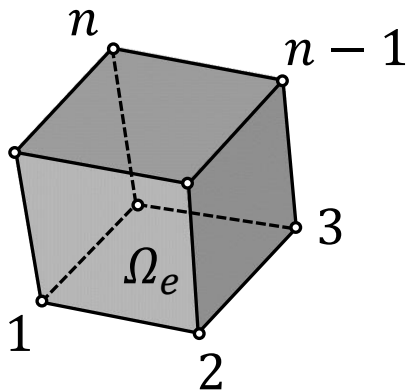
local stiffness matrix:

$$[k]_e = \int_{\Omega_e} [B]^T [D] [B] d\Omega_e$$

$n_e \times n_e \quad \Omega_e \quad n_e \times 6 \quad 6 \times 6 \quad 6 \times n_e$

Elastic strain energy in a finite element

local notation:



n – no. of nodes per FE

n_p – no. of nodal parameters per node

no. of degrees of freedom in FE:

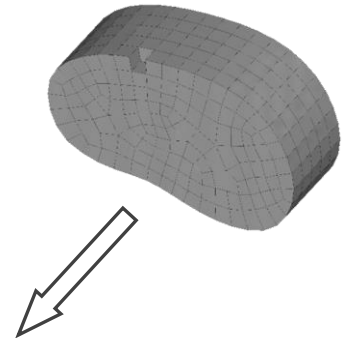
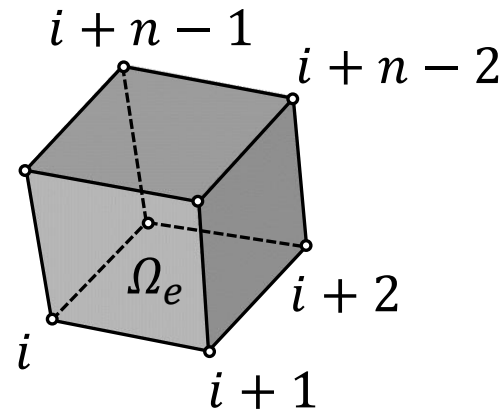
$$n_e = n \cdot n_p$$

$\{q\}_e$ - local vector of nodal parameters
 $n_e \times 1$

$$U_e = \frac{1}{2} \underset{1 \times n_e}{[q]_e} \underset{n_e \times n_e}{[k]_e} \underset{n_e \times 1}{\{q\}_e}$$

local stiffness matrix

global notation:



NON – no. of nodes

n_p – no. of nodal parameters per node

no. of degrees of freedom :

$$NDOF = NON \cdot n_p$$

$\{q\}$ - global vector of nodal parameters
 $NDOF \times 1$

$$U_e = \frac{1}{2} \underset{1 \times NDOF}{[q]} \underset{NDOF \times NDOF}{[k]_e^*} \underset{NDOF \times 1}{\{q\}}$$

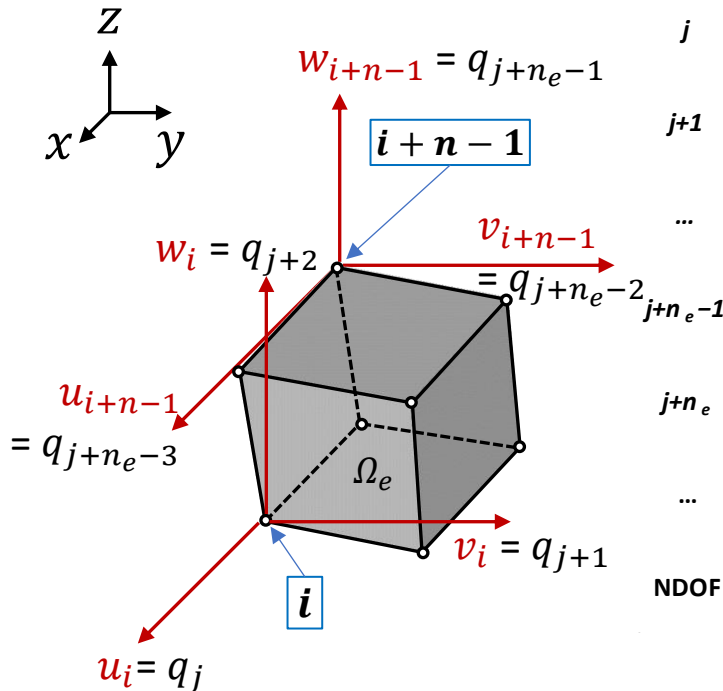
extended local stiffness matrix

Extended local stiffness matrix of a finite element

$$\{q\} = \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_j \\ \vdots \\ q_{NDOF} \end{Bmatrix}$$

$NDOF \times 1$

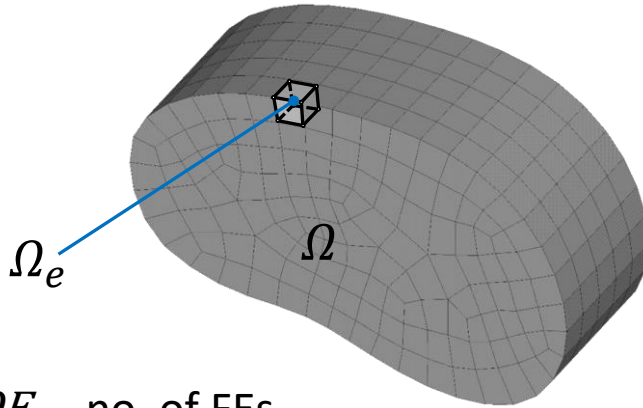
$$[k]_e^* =$$



	1	2	...	$j-1$	j	$j+1$	...	$j+n_e-1$	$j+n_e$	...	NDOF
1	0	0	...	0	0	0	...	0	0	...	0
2	0	0	...	0	0	0	...	0	0	...	0
...	...	...	...	0	0	0	...	0	0	...	0
$j-1$	0	0	0	0	0	0	...	0	0	...	0
j	0	0	0	0	k_{11}	k_{12}	...	k_{1n_e}	0	...	0
$j+1$	0	0	0	0	k_{21}	k_{22}	...	k_{2n_e}	0	...	0
...	...	...	...	...	...	...	...	...	0	...	0
$j+n_e-1$	0	0	0	0	k_{n_e1}	k_{n_e2}	...	$k_{n_en_e}$	0	...	0
$j+n_e$	0	0	0	0	0	0	0	0	0	...	0
...	...	...	...	...	...	...	...	...	...	...	0
NDOF	0	0	0	0	0	0	0	0	0	0	0

(assumed ascending order of components)

Elastic strain energy in a FE model. Global stiffness matrix



$$\Omega = \sum_{e=1}^{NOE} \Omega_e \rightarrow$$

$$U = \sum_{e=1}^{NOE} U_e$$

NOE – no. of FEs

NDOF – no. of degrees of freedom

$\{q\}$ - **global vector of nodal parameters**
 $NDOF \times 1$

elastic strain energy in a finite element model:

$$\begin{aligned} U &= \sum_{e=1}^{NOE} U_e = \sum_{e=1}^{NOE} \frac{1}{2} \cdot [q] \cdot [k]_e^* \cdot \{q\} = \frac{1}{2} [q] \cdot \sum_{e=1}^{NOE} [k]_e^* \cdot \{q\} = \\ &= \frac{1}{2} \cdot [q] \cdot [K] \cdot \{q\} \end{aligned}$$

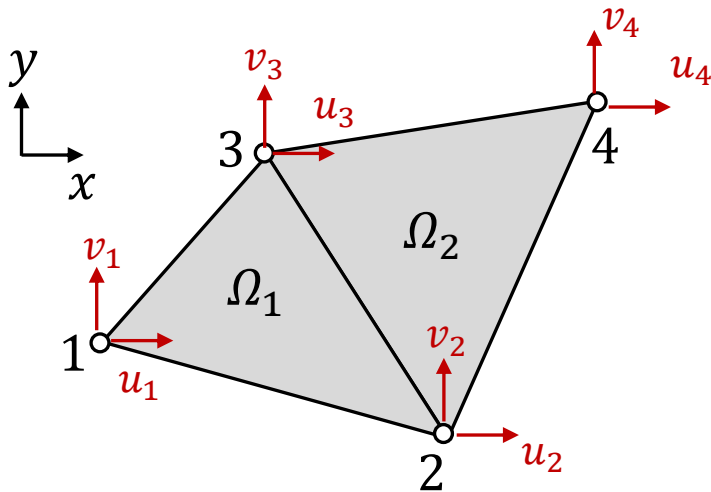
$1 \times NDOF$ $NDOF \times NDOF$ $NDOF \times 1$ $1 \times NDOF$ $NDOF \times NDOF$ $NDOF \times 1$
 $1 \times NDOF$ $NDOF \times NDOF$ $NDOF \times 1$

global stiffness matrix:

$$[K] = \sum_{e=1}^{NOE} [k]_e^*$$

$NDOF \times NDOF$

Example 4: global stiffness matrix of a 2D model with two 3-node triangles



global notation:

$$NOE = 2$$

$$NON = 4$$

$$n = 3$$

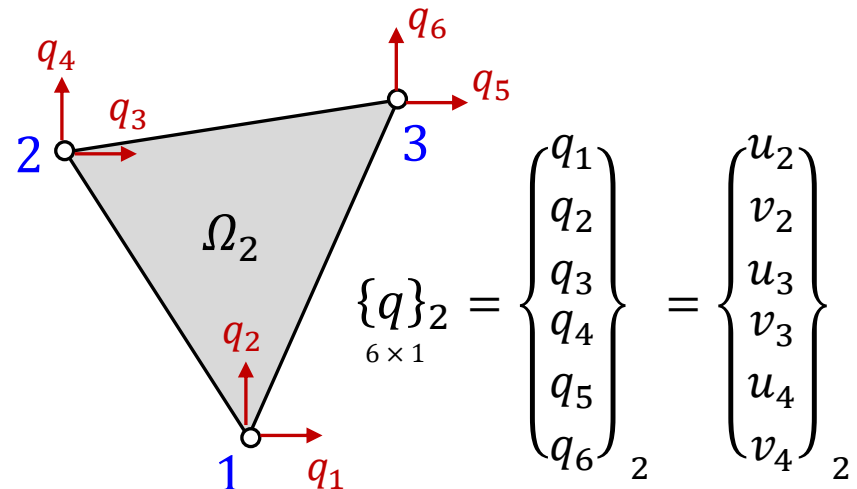
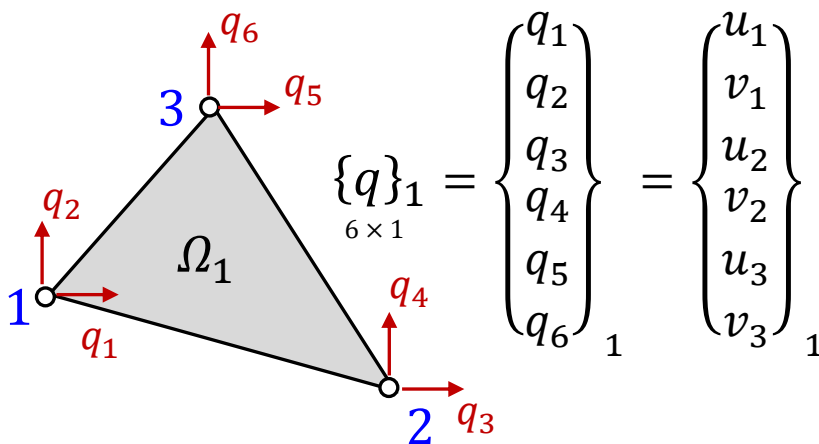
$$n_p = 2 \quad ; \quad (u, v)$$

$$n_e = n \cdot n_p = 6$$

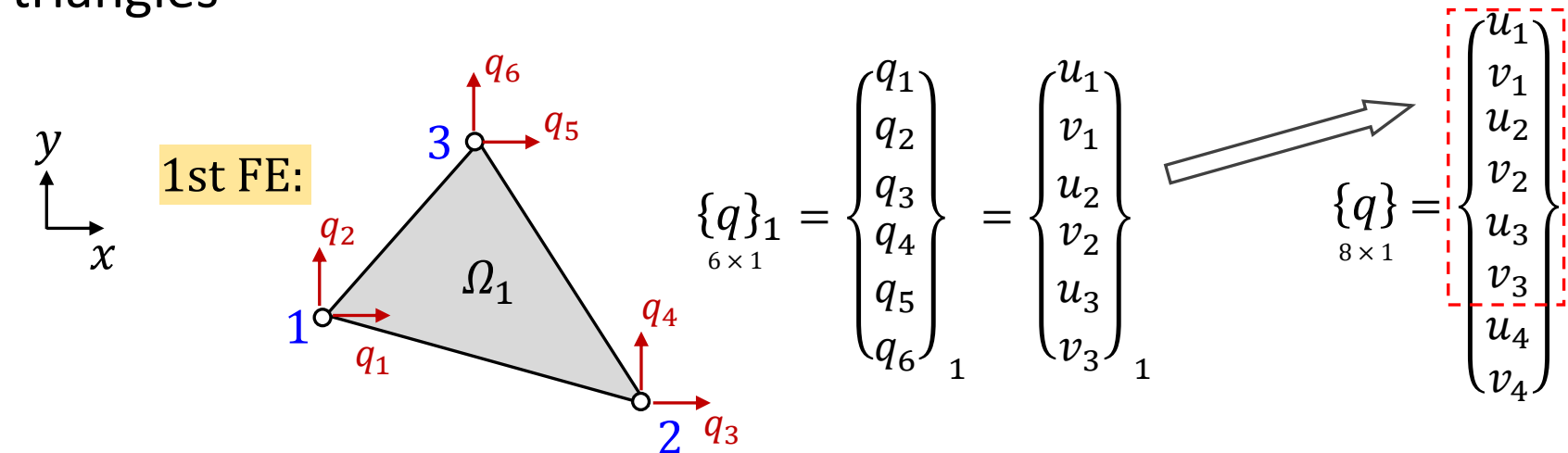
$$NDOF = NON \cdot n_p = 8$$

$$\{q\}_{8 \times 1} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

local notation:



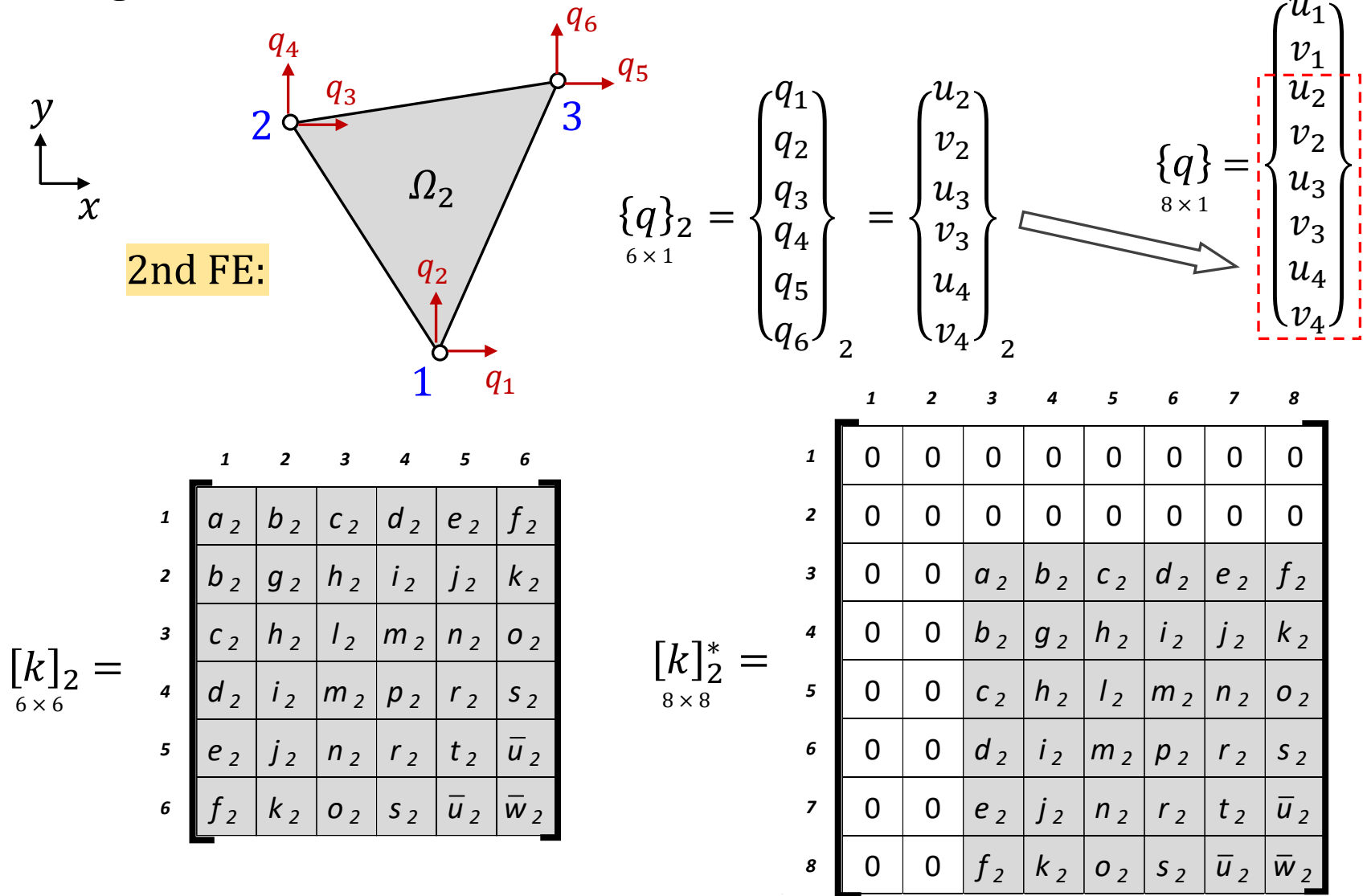
Example 4: global stiffness matrix of a 2D model with two 3-node triangles



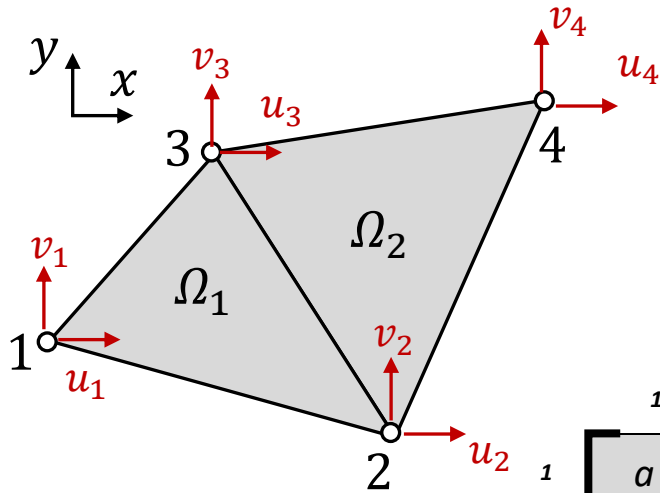
$$[k]_1 = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ b_1 & g_1 & h_1 & i_1 & j_1 & k_1 \\ c_1 & h_1 & l_1 & m_1 & n_1 & o_1 \\ d_1 & i_1 & m_1 & p_1 & r_1 & s_1 \\ e_1 & j_1 & n_1 & r_1 & t_1 & \bar{u}_1 \\ f_1 & k_1 & o_1 & s_1 & \bar{u}_1 & \bar{w}_1 \end{bmatrix} \end{matrix}$$

$$[k]_1^* = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} & \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 & 0 & 0 \\ b_1 & g_1 & h_1 & i_1 & j_1 & k_1 & 0 & 0 \\ c_1 & h_1 & l_1 & m_1 & n_1 & o_1 & 0 & 0 \\ d_1 & i_1 & m_1 & p_1 & r_1 & s_1 & 0 & 0 \\ e_1 & j_1 & n_1 & r_1 & t_1 & \bar{u}_1 & 0 & 0 \\ f_1 & k_1 & o_1 & s_1 & \bar{u}_1 & \bar{w}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

Example 4: global stiffness matrix of a 2D model with two 3-node triangles



Example 4: global stiffness matrix of a 2D model with two 3-node triangles



$$\{q\}_{8 \times 1} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

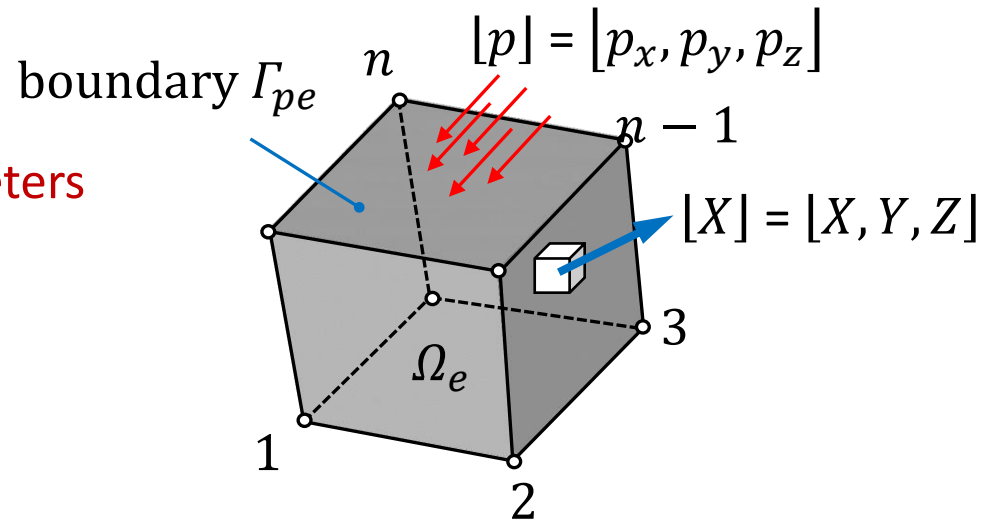
$$[K]_{8 \times 8} = [k]_1^* + [k]_2^* =$$

	1	2	3	4	5	6	7	8
1	a_1	b_1	c_1	d_1	e_1	f_1	0	0
2	b_1	g_1	h_1	i_1	j_1	k_1	0	0
3	c_1	h_1	$l_1 + a_2$	$m_1 + b_2$	$n_1 + c_2$	$o_1 + d_2$	e_2	f_2
4	d_1	i_1	$m_1 + b_2$	$p_1 + g_2$	$r_1 + h_2$	$s_1 + i_2$	j_2	k_2
5	e_1	j_1	$n_1 + c_2$	$r_1 + h_2$	$t_1 + l_2$	$\bar{u}_1 + m_2$	n_2	o_2
6	f_1	k_1	$o_1 + d_2$	$t_1 + l_2$	$\bar{u}_1 + m_2$	$\bar{w}_1 + p_2$	r_2	s_2
7	0	0	e_2	j_2	n_2	r_2	t_2	$\bar{u}_2$
8	0	0	f_2	k_2	o_2	s_2	$\bar{u}_2$	$\bar{w}_2$

Potential energy of loading in a finite element

$\{q\}_e$ - local vector of nodal parameters
 $n_e \times 1$

potential energy of loading
in a finite element :



$$W_e = \int_{\Omega_e} [X]_{1 \times 3} \{u\}_{3 \times 1} d\Omega_e + \int_{\Gamma_{pe}} [p]_{1 \times 3} \{u\}_{3 \times 1} d\Gamma_{pe} = \int_{\Omega_e} [X]_{1 \times 3} [N]_{3 \times n_e} \{q\}_e_{n_e \times 1} d\Omega_e + \int_{\Gamma_{pe}} [p]_{1 \times 3} [N]_{3 \times n_e} \{q\}_e_{n_e \times 1} d\Gamma_{pe} =$$

$$\begin{array}{c} \uparrow \\ \{u\} = [N] \{q\}_e \\ \begin{array}{ccc} 3 \times 1 & 3 \times n_e & n_e \times 1 \end{array} \end{array}$$

$$= \left(\int_{\Omega_e} [X]_{1 \times 3} [N]_{3 \times n_e} d\Omega_e + \int_{\Gamma_{pe}} [p]_{1 \times 3} [N]_{3 \times n_e} d\Gamma_{pe} \right) \{q\}_{n_e \times 1} = ([F^X]_{1 \times n_e} + [F^p]_{1 \times n_e}) \{q\}_{n_e \times 1} = [F]_{1 \times n_e} \{q\}_{n_e \times 1}$$

equivalent load vector:

$$\boxed{[F]_e = [F^X]_e + [F^p]_e}$$

Equivalent load vector

$$\underset{1 \times n_e}{[F]}_e = \underset{1 \times n_e}{[F^X]}_e + \underset{1 \times n_e}{[F^p]}_e$$

equivalent load vector due to mass forces:

$$\underset{1 \times n_e}{[F^X]}_e = \int_{\Omega_e} \underset{1 \times 3}{[X]} \underset{3 \times n_e}{[N]} d\Omega_e =$$

$$= \int_{\Omega_e} [X, Y, Z] \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_n & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \dots & 0 & N_n \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_n \end{bmatrix} d\Omega_e$$

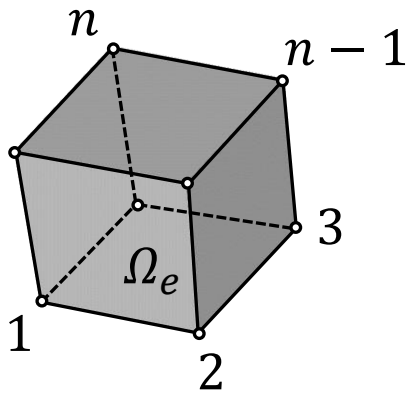
equivalent load vector due to surface load:

$$\underset{1 \times n_e}{[F^p]}_e = \int_{\Gamma_{pe}} \underset{1 \times 3}{[p]} \underset{3 \times n_e}{[N]} d\Gamma_{pe} =$$

$$= \int_{\Gamma_{pe}} [p_x, p_y, p_z] \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_n & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \dots & 0 & N_n \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_n \end{bmatrix} d\Gamma_{pe}$$

Potential energy of loading in a finite element

local notation:



n – no. of nodes per FE

n_p – no. of nodal parameters per node

no. of degrees of freedom in FE :

$$n_e = n \cdot n_p$$

$\{q\}_e$ - local vector of nodal parameters
 $n_e \times 1$

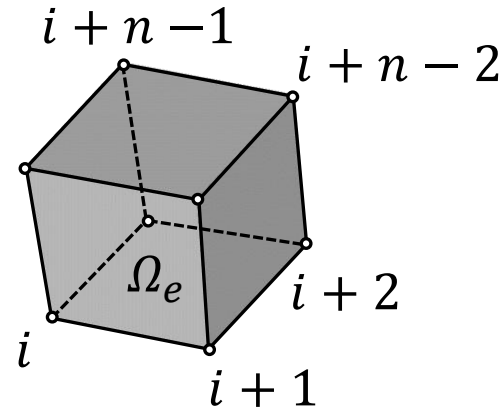
$$W_e = [q]_e \{F\}_e$$

$1 \times n_e \quad n_e \times 1$



equivalent load vector

global notation:



NON – no. of nodes

n_p – no. of nodal parameters per node

no. of degrees of freedom :

$$NDOF = NON \cdot n_p$$

$\{q\}$ - global vector of nodal parameters
 $NDOF \times 1$

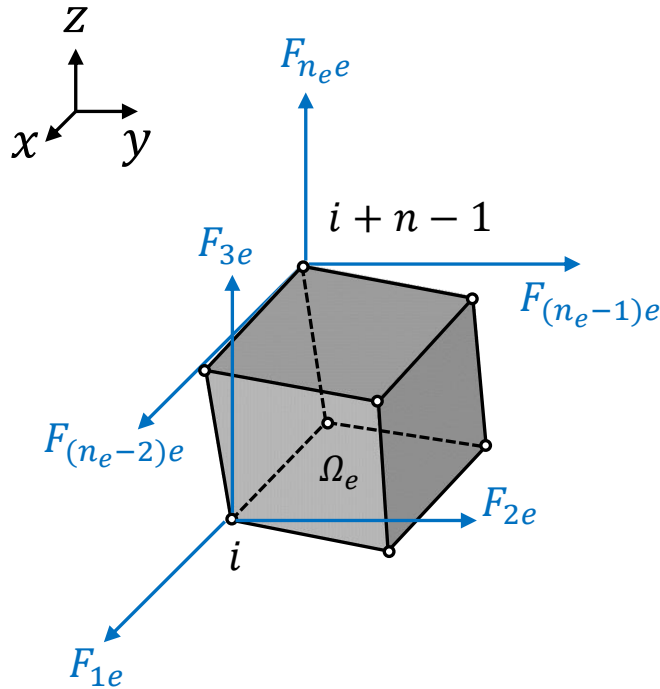
$$W_e = [q] \cdot \{F\}_e^*$$

$1 \times NDOF \quad NDOF \times 1$



extended equivalent load vector

Extended equivalent load vector in a finite element



extended equivalent load vector:

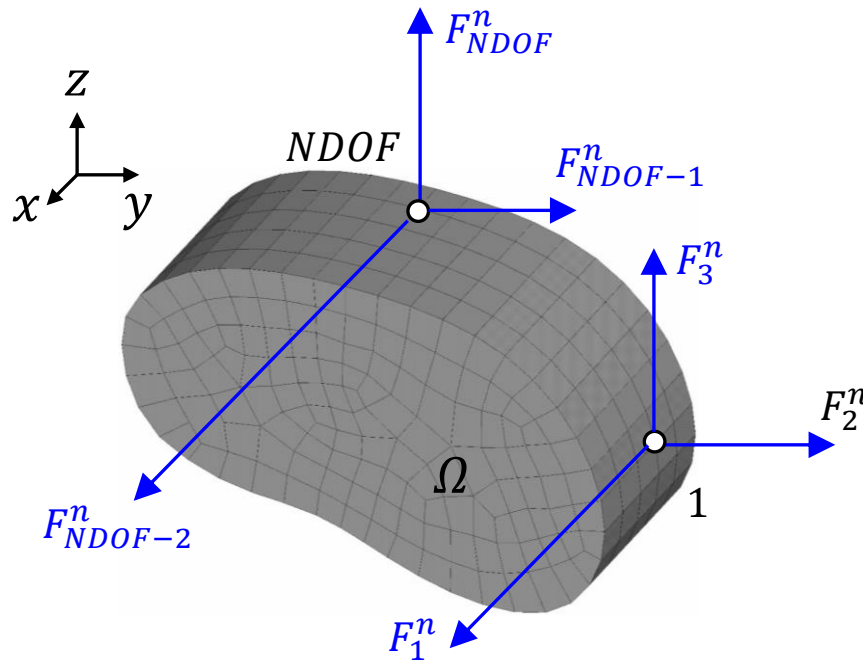
equivalent load vector:

$$\{F\}_e = \begin{Bmatrix} F_{1e} \\ F_{2e} \\ F_{3e} \\ \dots \\ F_{(n_e-2)e} \\ F_{(n_e-1)e} \\ F_{n_e e} \end{Bmatrix}_{n_e \times 1}$$

$$\{F\}_e^* = \begin{Bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ F_{1e} \\ F_{2e} \\ \dots \\ F_{n_e e} \\ 0 \\ \dots \\ 0 \end{Bmatrix}_{NDOF \times 1}$$

(assumed ascending order of components)

Forces applied directly at nodes. Potential energy of nodal loads



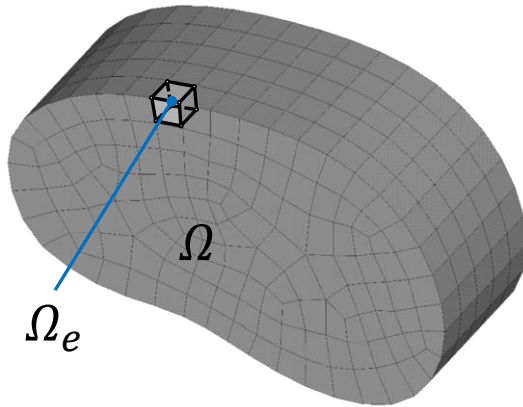
nodal load vector:

$$\{F\}_{NDOF \times 1}^n = \begin{Bmatrix} F_1^n \\ F_2^n \\ F_3^n \\ \dots \\ F_{NDOF-2}^n \\ F_{NDOF-1}^n \\ F_{NDOF}^n \end{Bmatrix}$$

potential energy of nodal loads:

$$W^n = [q]_{1 \times NDOF} \cdot \{F\}_{NDOF \times 1}^n$$

Potential energy of loading in a FE model. Global load vector



NOE – no. of FEs

$NDOF$ – no. of degrees of freedom

potential energy of element loads:

$$\Omega = \sum_{e=1}^{NOE} \Omega_e \rightarrow W^e = \sum_{e=1}^{NOE} W_e$$

potential energy
of nodal loads

potential energy of loading in a finite element model:

$$W = W^e + W^n$$

$$W = \sum_{e=1}^{NOE} W_e + W^n = \sum_{e=1}^{NOE} [q] \cdot \{F\}_e^* + [q] \cdot \{F\}^n = [q] \cdot \left(\sum_{e=1}^{NOE} \{F\}_e^* + \{F\}^n \right)$$

$1 \times NDOF$ $NDOF \times 1$ $1 \times NDOF$ $NDOF \times 1$ $1 \times NDOF$ $NDOF \times 1$ $NDOF \times 1$

$$= [q] \cdot (\{F\}^e + \{F\}^n) \rightarrow W = [q] \cdot \{F\}$$

$1 \times NDOF$ $NDOF \times 1$ $NDOF \times 1$ $1 \times NDOF$ $NDOF \times 1$

global load vector
of element loads

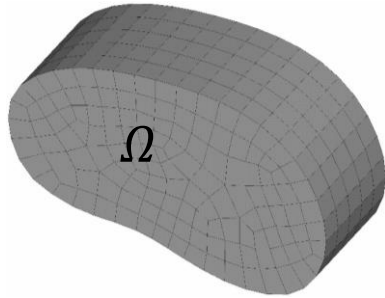
nodal load vector

global load vector:

$$\{F\} = \{F\}^e + \{F\}^n$$

$NDOF \times 1$ $NDOF \times 1$ $NDOF \times 1$

Total potential energy in a FE model. Set of linear equations



Total potential energy of the entire model:

$$V = U - W = \frac{1}{2} \cdot \underset{1 \times NDOF}{[q]} \cdot \underset{NDOF \times NDOF}{[K]} \cdot \underset{NDOF \times 1}{\{q\}} - \underset{1 \times NDOF}{[q]} \cdot \underset{NDOF \times 1}{\{F\}}$$

$$\underset{NDOF \times 1}{\{q\}} = ?$$

NOE – no. of FEs

$NDOF$ – no. of degrees of freedom

$$V \rightarrow \min$$

$$\frac{\partial V}{\partial q_j} = 0 \rightarrow$$

$$\underset{NDOF \times NDOF}{[K]} \cdot \underset{NDOF \times 1}{\{q\}} = \underset{NDOF \times 1}{\{F\}}$$



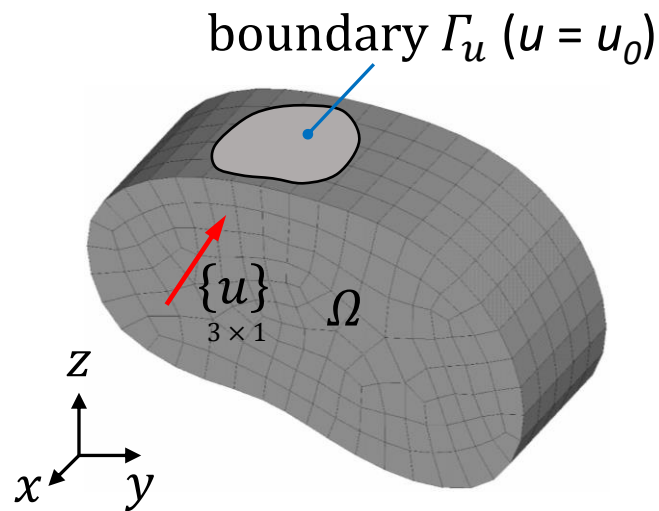
set of linear algebraic equations

$$\underset{NDOF \times NDOF}{\det ([K])} = 0$$



Set of FE equations with boundary conditions

The displacement field $\{u\}$ that represents solution of the problem fulfils displacement boundary conditons on Γ_u and minimizes the total potential energy V .



$NDOF$ – no. of degrees of freedom

NOF – no. of known degrees of freedom on Γ_u

N – number of unknown degrees of freedom:

$$N = NDOF - NOF$$

$$\begin{array}{ccccccc} [K] & \rightarrow & [K] & ; & \{q\} & \rightarrow & \{q\} & ; & \{F\} & \rightarrow & \{F\} \\ NDOF \times NDOF & & N \times N & & NDOF \times 1 & & N \times 1 & & NDOF \times 1 & & N \times 1 \end{array}$$

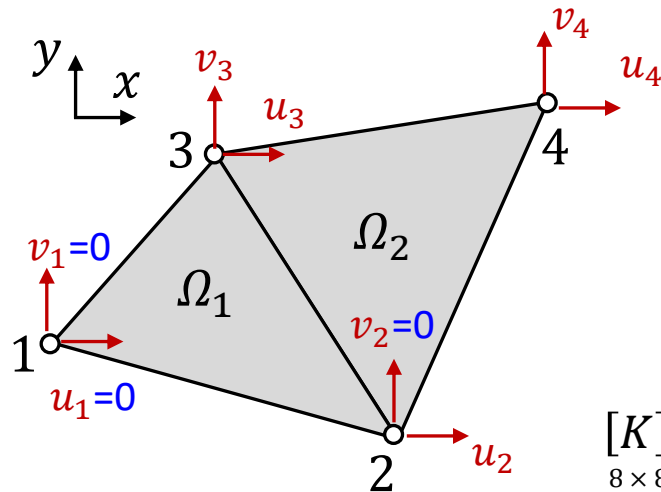
$$\begin{array}{ccc} [K] \cdot \{q\} & = & \{F\} \\ N \times N & & N \times 1 \end{array}$$

$$\det ([K]) \neq 0$$

$N \times N$

linear set of algebraic equations with boundary conditions

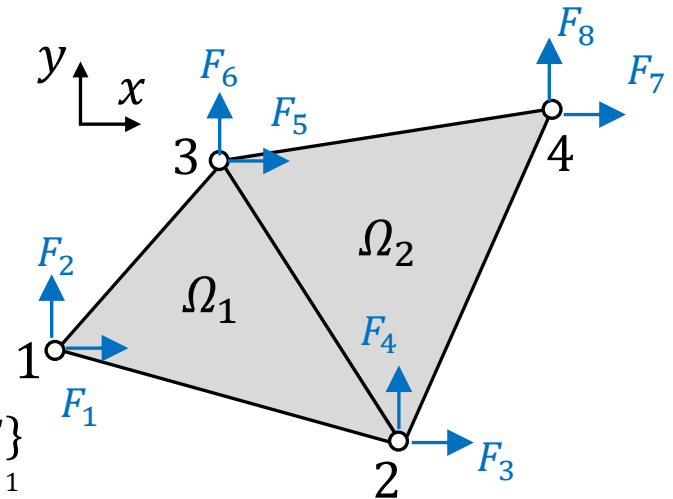
Example 5. Boundary conditions for 2D problem. FE model with two 3-node triangles



$NDOF = 8$

$NOF = 3$

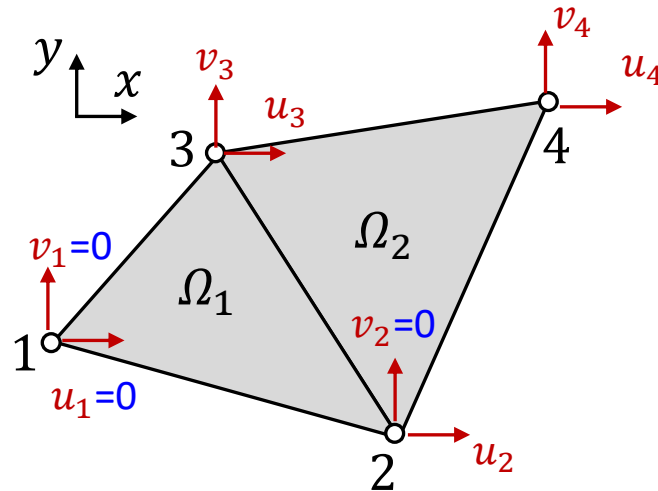
$$[K]_{8 \times 8} \cdot \{q\}_{8 \times 1} = \{F\}_{8 \times 1}$$



	1	2	3	4	5	6	7	8
1	a_1	b_1	c_1	d_1	e_1	f_1	0	0
2	b_1	g_1	h_1	i_1	j_1	k_1	0	0
3	c_1	h_1	$l_1 + a_2$	$m_1 + b_2$	$n_1 + c_2$	$o_1 + d_2$	e_2	f_2
4	d_1	i_1	$m_1 + b_2$	$p_1 + g_2$	$r_1 + h_2$	$s_1 + i_2$	j_2	k_2
5	e_1	j_1	$n_1 + c_2$	$r_1 + h_2$	$t_1 + l_2$	$\bar{u}_1 + m_2$	n_2	o_2
6	f_1	k_1	$o_1 + d_2$	$t_1 + l_2$	$\bar{u}_1 + m_2$	$\bar{w}_1 + p_2$	r_2	s_2
7	0	0	e_2	j_2	n_2	r_2	t_2	$\bar{u}_2$
8	0	0	f_2	k_2	o_2	s_2	$\bar{u}_2$	$\bar{w}_2$

$$\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 \\ v_2 = 0 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix}$$

Example 5. Boundary conditions for 2D problem. FE model with two 3-node triangles

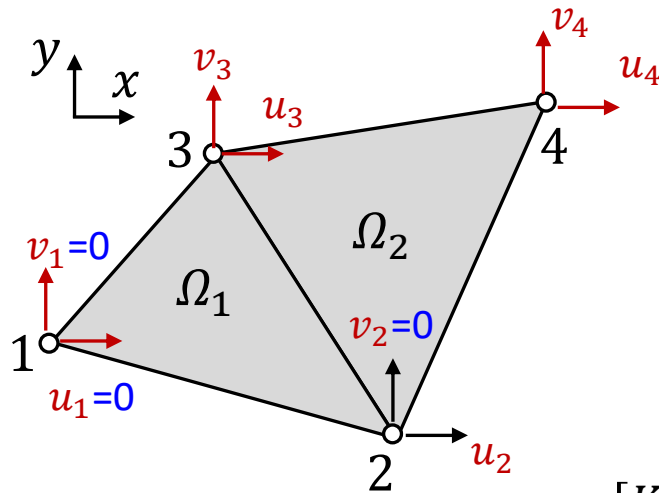


$$\underset{8 \times 8}{[K]} \cdot \underset{8 \times 1}{\{q\}} = \underset{8 \times 1}{\{F\}}$$

	1	2	3	4	5	6	7	8
1	a_1	b_1	c_1	d_1	e_1	f_1	0	0
2	b_1	g_1	h_1	i_1	j_1	k_1	0	0
3	c_1	h_1	l_1+a_2	m_1+b_2	n_1+c_2	o_1+d_2	e_2	f_2
4	d_1	i_1	m_1+b_2	p_1+g_2	r_1+h_2	s_1+i_2	j_2	k_2
5	e_1	j_1	n_1+c_2	r_1+h_2	t_1+l_2	$\bar{u}_1+m_2$	n_2	o_2
6	f_1	k_1	o_1+d_2	t_1+l_2	$\bar{u}_1+m_2$	$\bar{w}_1+p_2$	r_2	s_2
7	0	0	e_2	j_2	n_2	r_2	t_2	$\bar{u}_2$
8	0	0	f_2	k_2	o_2	s_2	$\bar{u}_2$	$\bar{w}_2$

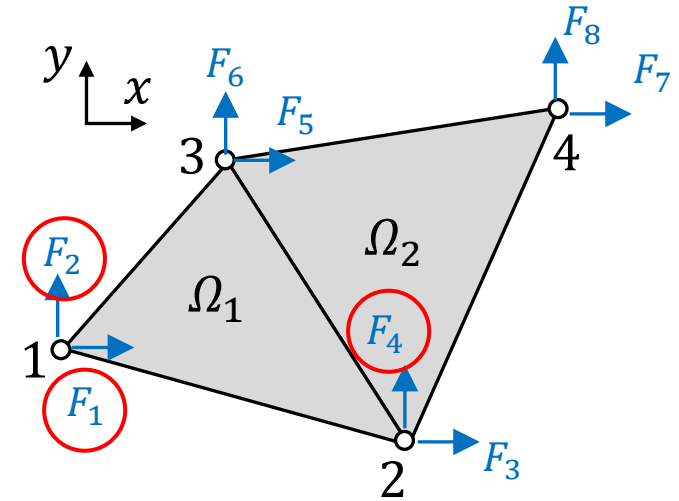
$$\begin{Bmatrix} u_1=0 \\ v_1=0 \\ u_2 \\ v_2=0 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix}$$

Example 5. Boundary conditions for 2D problem. FE model with two 3-node triangles



$$N = 8 - 3 = 5$$

$$\underset{5 \times 5}{[K]} \cdot \underset{5 \times 1}{\{q\}} = \underset{5 \times 1}{\{F\}}$$



$l_1 + a_2$	$n_1 + c_2$	$o_1 + d_2$	e_2	f_2
$n_1 + c_2$	$t_1 + l_2$	$u_1 + m_2$	n_2	o_2
$o_1 + d_2$	$u_1 + m_2$	$w_1 + p_2$	r_2	s_2
e_2	n_2	r_2	t_2	$\bar{u}_2$
f_2	o_2	s_2	$\bar{u}_2$	$\bar{w}_2$

$$\begin{Bmatrix} u_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix}$$

↑

linear set of algebraic equations with boundary conditions

Solution of a set of FE equations with boundary conditions

$$\underset{N \times N}{[K]} \cdot \underset{N \times 1}{\{q\}} = \underset{N \times 1}{\{F\}} \quad \rightarrow \quad \underset{N \times N}{\det ([K])} \neq 0 \quad \rightarrow \quad \underset{N \times 1}{\{q\}} = \underset{N \times N}{[K]}^{-1} \underset{N \times 1}{\{F\}}$$

DOF solution: $\underset{NDOF \times 1}{\{q\}}$

Element solution (ES): $\underset{6 \times 1}{\{\varepsilon\}} = \underset{6 \times n_e}{[B]} \underset{n_e \times 1}{\{q\}_e} \quad ; \quad \underset{6 \times 1}{\{\sigma\}} = \underset{6 \times 6}{[D]} \underset{6 \times 1}{\{\varepsilon\}} = \underset{6 \times 6}{[D]} \underset{6 \times n_e}{[B]} \underset{n_e \times 1}{\{q\}_e}$

$\uparrow$ $\uparrow$
 strain in a finite element stress in a finite element

Nodal solution (NS):

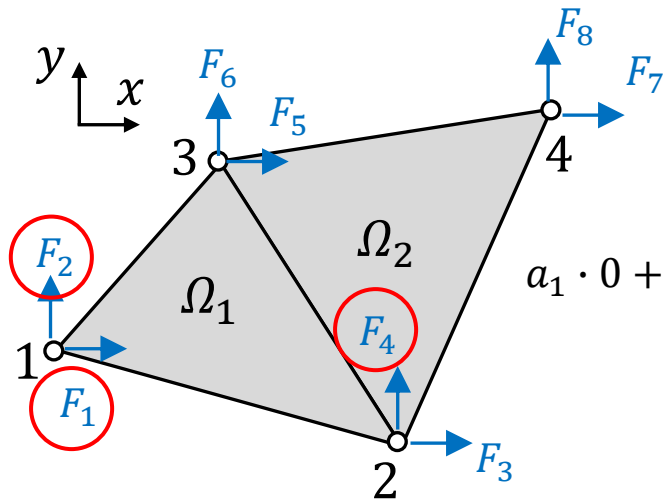
$$(NS)_i = \frac{\sum_{e=1}^k (ES)_{ei}}{k}$$

$(NS)_i$ – averaged nodal solution at node (i)

$(ES)_{ei}$ – element solution in element (e) and at node (i)

k – no. of elements adjacent to node (i)

Example 6. Reactions calculation for 2D problem. FE model with two 3-node triangles



$$[K] \cdot \{q\} = \{F\}$$

8×8 8×1 8×1
 $\uparrow$
known

$$\boxed{} \cdot \boxed{} = F_1$$

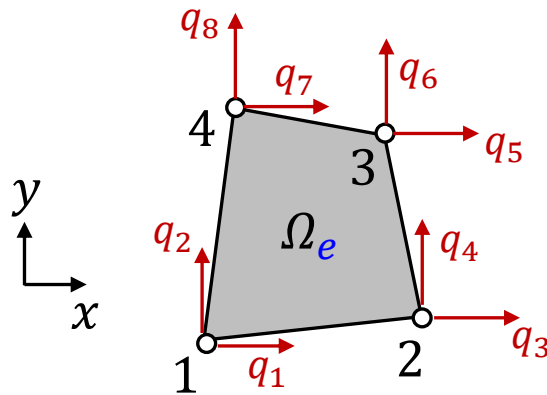
$$a_1 \cdot 0 + b_1 \cdot 0 + c_1 \cdot u_2 + d_1 \cdot 0 + e_1 \cdot u_3 + f_1 \cdot v_3 + 0 \cdot u_4 + 0 \cdot v_4 = F_1$$

$$\boxed{} \cdot \boxed{} = F_2 ; \quad \boxed{} \cdot \boxed{} = F_4$$

	1	2	3	4	5	6	7	8
1	a_1	b_1	c_1	d_1	e_1	f_1	0	0
2	b_1	g_1	h_1	i_1	j_1	k_1	0	0
3	c_1	h_1	$l_1 + a_2$	$m_1 + b_2$	$n_1 + c_2$	$o_1 + d_2$	e_2	f_2
4	d_1	i_1	$m_1 + b_2$	$p_1 + g_2$	$r_1 + h_2$	$s_1 + i_2$	j_2	k_2
5	e_1	j_1	$n_1 + c_2$	$r_1 + h_2$	$t_1 + l_2$	$\bar{u}_1 + m_2$	n_2	o_2
6	f_1	k_1	$o_1 + d_2$	$t_1 + l_2$	$\bar{u}_1 + m_2$	$\bar{w}_1 + p_2$	r_2	s_2
7	0	0	e_2	j_2	n_2	r_2	t_2	$\bar{u}_2$
8	0	0	f_2	k_2	o_2	s_2	$\bar{u}_2$	$\bar{w}_2$

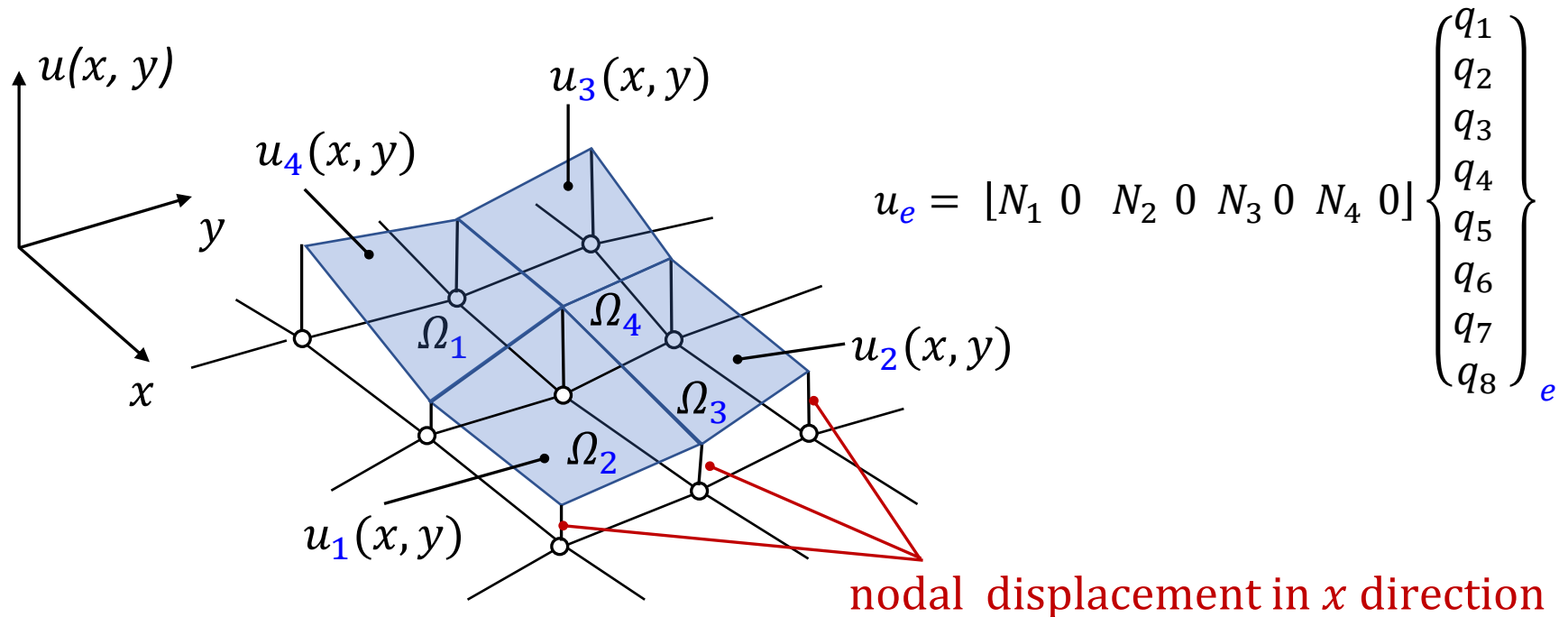
$$\left\{ \begin{array}{l} u_1 = 0 \\ v_1 = 0 \\ u_2 \\ v_2 = 0 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{array} \right\} = \left\{ \begin{array}{l} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{array} \right\}$$

Example 7. DOF solution $u(x,y)$ for 2D problem. FE model with 4-node quadrilateral elements



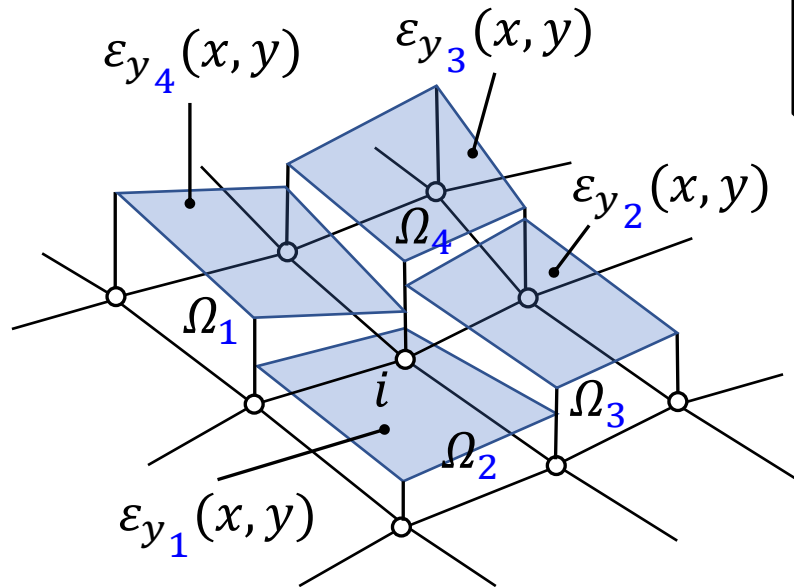
$$\{u\}_{2 \times 1} = [N]_{2 \times 8} \{q\}_{8 \times 1}_e$$

$u_e(x, y)$ – displacement in x direction

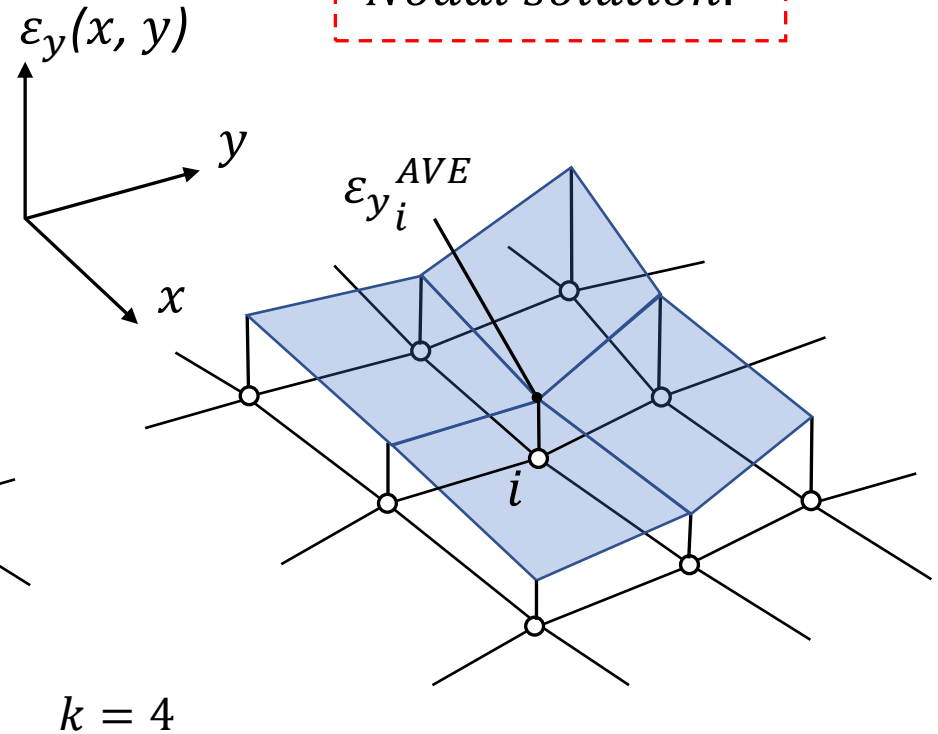


Example 8. Strain component $\varepsilon_y(x,y)$ for 2D problem. FE model with 4-node quadrilateral elements

Element solution:

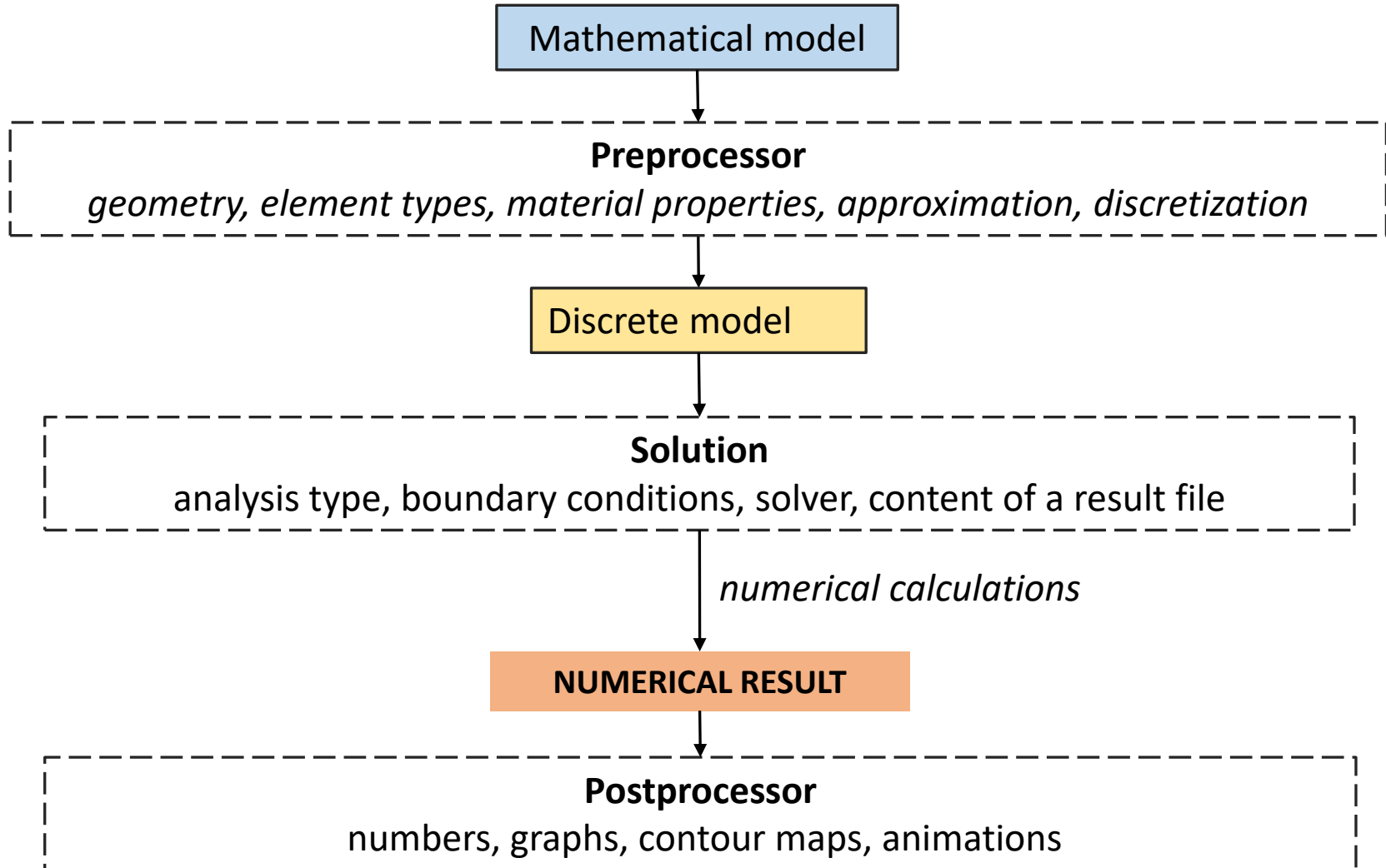


Nodal solution:

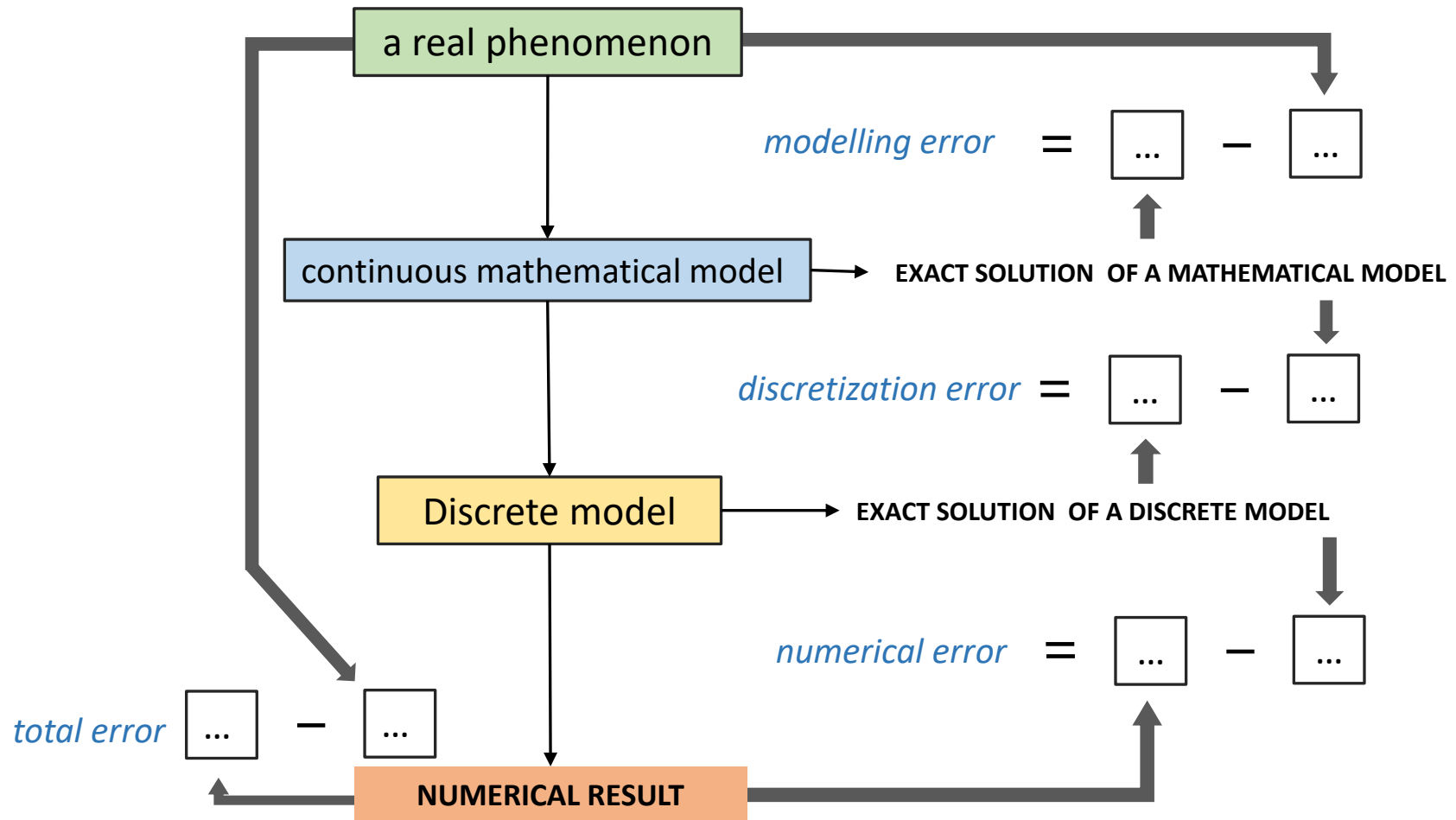


$$\varepsilon_{y_i}^{AVE} = \frac{\varepsilon_{y_1}(x_i, y_i) + \varepsilon_{y_2}(x_i, y_i) + \varepsilon_{y_3}(x_i, y_i) + \varepsilon_{y_4}(x_i, y_i)}{4}$$

FE modelling – basic steps



Accuracy of FEM calculations



$total\ error = modelling\ error + discretization\ error + numerical\ error$

$modelling\ error \approx discretization\ error \approx numerical\ error \rightarrow min$